FLEXIBLE PARTIALLY STABLE ALGEBRAS[1]

BY
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I. **Introduction.** A flexible power associative algebra A is a power associative algebra satisfying the identity

$$(x, y, x) = 0 \qquad \text{for all } x, y \text{ in } A, \tag{1.1}$$

where

$$(a, b, c) = (ab)c - a(bc).$$

This identity is equivalent to either one of the following identities provided the characteristic of the ground field F is different from two.

$$(x, y, z) + (z, y, x) = 0 \quad \text{for all } x, y, z \text{ in } A \tag{1.1a}$$

$$(xy)z + (zy)x = x(yz) + z(yx). \tag{1.1b}$$

Let A be any power associative algebra, then A defines a commutative power associative algebra A^+, where A^+ has the same vector space as A and A^+ has as its multiplication the composition $a \cdot b = 1/2(ab + ba)$. Throughout this paper we shall use $x \cdot y$ to indicate the product in A^+ to distinguish from the product xy in A and the term algebra to mean finite dimensional algebra.

In the sequel we may need a useful identity

$$(x \cdot y)z + (z \cdot y)x = x \cdot (yz) + z \cdot (yx) \tag{1.2}$$

which is obtained from adding $(yx)z + (yz)x$ to both sides of (1.1b).

Let u be an idempotent of A. One may obtain the Pierce decomposition of A with respect to this idempotent u as:

$$A = A_u(1) + A_u(\tfrac{1}{2}) + A_u(0) \qquad \text{(vector space direct sum)}$$

where $A_u(\lambda)$, $\lambda = 1, \frac{1}{2}, 0$, is a subspace of A such that x is in $A_u(\lambda)$ if and only if $ux + xu = \lambda x$.

It is well known ([1]) that if A is a flexible power associative algeara, the $A_u(1)$ and $A_u(0)$ are orthogonal subalgebras of A, and $A_u(1)A_u(\frac{1}{2}) \subseteq A_u(\frac{1}{2}) + A_u(0)$;

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$A_u(\frac{1}{2})A_u(1) \subseteq A_u(\frac{1}{2}) + A_u(0)$; $A_u(0)A_u(\frac{1}{2}) \subseteq A_u(\frac{1}{2}) + A_u(1)$; $A_u(\frac{1}{2})A_u(0) \subseteq A_u(\frac{1}{2}) + A_u(1)$; $A_u(\frac{1}{2}) \cdot A_u(\frac{1}{2}) \subseteq A_u(1) + A_u(0)$.

A flexible algebra will be called a u-stable algebra if $A_u(\lambda)A_u(\frac{1}{2}) \subseteq A_u(\frac{1}{2}), \lambda = 0, 1$. A will be called a stable algebra if it is u-stable with respect to every idempotent u in A. A flexible algebra will be called a u-nilstable algebra if $A_u(\lambda)A_u(\frac{1}{2}) \subseteq A_u(\frac{1}{2}) + N_{1-\lambda}$, $\lambda = 0, 1$, where N_1 and N_0 are nilsubalgebras of $A_u(1)$ and $A_u(0)$ respectively, ([8], [12]) and will be called a nilstable if it is u-nilstable with respect to every idempotent u.

It is known that every simple, flexible, strictly power associative algebra has a unity element 1 ([11]). An idempotent u is called a primitive idempotent of A if in the Pierce decomposition, $A_u(1)$ contains no other idempotent besides u. The unity element 1 is a sum of pairwise primitive orthogonal idempotents $u_1, u_2, \cdots, u_t$, where t is a unique positive integer and is called the degree of A. A simple, flexible, strictly power associative algebra A over a field F of characteristic not equal to 2 or 3 is an algebra of the following type: (1) a commutative Jordan algebra, (2) a quasi-associative algebra, (3) an algebra of degree two, (4) an algebra of degree one. Among these classes, the first two have been quite thoroughly studied, and the last one has been recently studied by Kokoris and Kleinfeld [10].

Two special classes of flexible algebras of degree two have been studied recently, ([9], [12]). It is known that every simple flexible power associative nilstable (stable) algebra of degree two over an algebraically closed field F of characteristic not equal 2, 3, 5 is a J-simple algebra [12].

In this paper we shall investigate the structure of a class of simple flexible power associative algebras of degree two which are stable with respect to an idempotent u. It is easy to show $v = 1 - u$ is also an idempotent of A. In fact, $A_u(\lambda) = A_u(1 - \lambda)$; $\lambda = 0, \frac{1}{2}, 1$.

II. **Simplicities of A and A^+.** Throughout this section the algebra A is assumed to be of degree two and u-stable with respect to an idempotent u. The notations A_λ will be reserved for $A_u(\lambda), \lambda = 0, \frac{1}{2}, 1$ the component vector spaces in the Pierce decomposition of A.

THEOREM 2.1. *Let A be a flexible u-stable power associative algebra of degree two over an algebraically closed field F of characteristic not 2, 3 or 5. Then the algebra A is simple if, and only if, A^+ is simple.*

This theorem is obviously true in one direction, for any (two-sided) ideal in A is also an ideal in A^+. Consequently, if A^+ is simple, so is A. In order to prove the implication in the other direction, the following preparatory lemmas will be useful. The proofs of some of these lemmas are straight forward and we will omit them.

LEMMA 2.1. *Let $J = M_1 + M_{1/2} + M_0$ be an A^+-ideal, where $M_\lambda \subseteq A_\lambda$; and let $a = a_1 + a_{1/2} + a_0$ be an element in J, then a_λ is in J, $\lambda = 0, \frac{1}{2}, 1$. Furthermore M_1, M_0 are ideals of A_1^+ and A_0^+ respectively.*

Let $x = u + r$ be an element of A_1^+. Where r is nilpotent and $r^t = 0$. Then

$$(u + r)\cdot(u - r + r^2 + \cdots + (-1)^{t-1}r^{t-1}) = u,$$

the element $x_u^{-1} = u - r + r^2 + \cdots + (-1)^{t-1}r^{t-1}$ is in A_1^+ and is the inverse of x. We shall call x a nonsingular element of A_1^+. A nonsingular element of A_0^+ is defined in similar fashion.

LEMMA 2.2. *If both M_1 and M_0 contain nonsingular elements, then $J = A^+$. If only M_1 contains nonsingular elements, but M_0 does not contain nonsingular elements, then $A_1^+ = M_1$ and $J' = N_1 + M_{1/2} + M_0$ is also a proper ideal of A^+.*

We ,then, shall assume that if J is a proper A^+-ideal then $J = M_1 + M_{1/2} + M_0$, where $M_\lambda \subseteq N_\lambda$, $\lambda = 0, 1$. Let $N = N_1 + N_0$ and let A_N be the set of all quantities y in $A_{1/2}$ such that for every z in $A_{1/2}$, $y \cdot z$ is in N. It is easy to show that A_N is a subspace of $A_{1/2}$. The subspace A_N will be called the singular subspace of $A_{1/2}$.

LEMMA 2.3. *If J is a proper ideal of A^+, and $J = M_1 + M_{1/2} + M_0$ then $M_{1/2} \subseteq A_N$.*

Proof. It suffices to show that if y is in $M_{1/2}$, then $y \cdot z$ is in N for all z in $A_{1/2}$. Suppose this is not true, then there exists an element z in $A_{1/2}$ such that $y \cdot z = \alpha + n$, and $\alpha \neq 0$. But y is in $M_{1/2} \subseteq J$ so $\alpha + n = y \cdot z$ is in J. Hence $1 = \alpha^{-2}(\alpha + n)\cdot(\alpha - n + n^2 + \cdots)$ is in J, so $J = A^+$. This contradicts our assumption.

LEMMA 2.4. *Let A_N be the singular subspace of an algebra A and $N = N_1 + N_0$, then $N \cdot A_N \subseteq A_N$.*

Proof. It is well known that a commutative algebra A of characteristics not 2, 3 or 5 is power associative if and only if the identity

$$\begin{aligned} &4(w\cdot x)\cdot(y\cdot z) + 4(w\cdot y)\cdot(x\cdot z) + 4(w\cdot z)\cdot(x\cdot v) \\ &= w\cdot[(x\cdot y)\cdot z + (y\cdot z)\cdot x + (z\cdot x)\cdot y] + x\cdot[(y\cdot z)\cdot w + (z\cdot w)\cdot v + (w\cdot y)\cdot z] \\ &+ y\cdot[(z\cdot w)\cdot x + (w\cdot x)\cdot z + (x\cdot z)\cdot w] + z\cdot[(w\cdot x)\cdot y + (x\cdot y)\cdot w + (y\cdot w)\cdot x] \end{aligned} \tag{2.1}$$

is satisfied by any elements x, y, x, w of A ([1]). Let x be in N_1, y be in A_N, z be in $A_{1/2}$ and $w = u$. We have

$$\begin{aligned} &4x\cdot(y\cdot z) + 2y\cdot(x\cdot z) + 2z\cdot(x\cdot y) \\ &= x\cdot(y\cdot z) + u\cdot[(x\cdot y)\cdot z + (x\cdot z)\cdot y] + 2x\cdot(y\cdot z) + 2y\cdot(x\cdot z) + 2z\cdot(x\cdot y) \end{aligned} \tag{2.2}$$

using the facts $u\cdot x = x$, $u\cdot y = \frac{1}{2}y$, and $u\cdot z = \frac{1}{2}z$. Since $y\cdot z$ is in $N_1 + N_0$, $x\cdot[(y\cdot z)\cdot u] = x\cdot(y\cdot z) = u\cdot[x\cdot(y\cdot z)]$ is in N_1. We have

$$x\cdot(y\cdot z) = u\cdot[(x\cdot y)\cdot z + (x\cdot z)\cdot y]. \tag{2.3}$$

Let $(x\cdot z)\cdot y = m$, $(x\cdot y)\cdot z = \alpha + n$, where m, n are in N. Then, $u\cdot[(x\cdot y)\cdot z + (x\cdot z)\cdot y] = \alpha u + n_1 + m_1$ ($n_1 = n\cdot u$, $m_1 = m\cdot u$). However x is in N_1, so $x\cdot(y\cdot z)$ is in N_1, thus $\alpha = 0$ and $(x\cdot y)\cdot z = n_1$ is in N. It yields $N_1\cdot A_N \subseteq A_N$. Similarly $N_0\cdot A_N \subseteq A_N$.

Corollary 1. *For every a in $C = A_1 + A_0$ and y in A_N, $a\cdot y$ is in A_N that is, $C\cdot A_N \subseteq A_N$.*

Lemma 2.5. *Let $J = M_1 + M_{1/2} + M_0$ be a proper A^+-ideal; then,*

$$M_{1/2}A_{1/2} \subseteq M_1 + A_{1/2} + M_0,\quad A_{1/2}M_{1/2} \subseteq M_1 + A_{1/2} + M_0$$

and

$$(M_{1/2}A_{1/2})_{1/2} = (A_{1/2}M_{1/2})_{1/2}.$$

Proof. It will suffice to show the following five statements.

$$(M_{1/2}A_{1/2})_1 \subseteq M_1; \tag{2.3.a}$$

$$(A_{1/2}M_{1/2})_1 \subseteq M_1; \tag{2.3.b}$$

$$(M_{1/2}A_{1/2})_0 \subseteq M_0; \tag{2.3.c}$$

$$(A_{1/2}M_{1/2})_0 \subseteq M_0; \tag{2.3.d}$$

$$(M_{1/2}A_{1/2})_{1/2} = (A_{1/2}M_{1/2})_{1/2}. \tag{2.3.e}$$

Obviously $(A_{1/2}M_{1/2})_{1/2} = (M_{1/2}A_{1/2})_{1/2}$, since $A_{1/2}M_{1/2} \subseteq A_{1/2}\cdot M_{1/2} + M_{1/2}A_{1/2}$ and $A_{1/2}\cdot M_{1/2} = M_1 + M_0$ follows from the fact that J is an A^+-ideal. Let y be in $M_{1/2}$, z be in $A_{1/2}$, $y\cdot z = m_1 + m_0$, $yz = \alpha u + t_1 + a_{1/2} + \beta v + t_0$ and $zy = -\alpha u + s_1 - a_{1/2} - \beta v + s_0$ where m_i is in M_i, s_i, t_i, a_i are in A_i, and α, β, are in F. Also let $y^* = uy - \frac{1}{2}y = \frac{1}{2}y - yu$, for all y in $A_{1/2}$. [Since $2(u\cdot z)y = zy = -\alpha u + s_1 - a_{1/2} - \beta v + s_0$, $2y\cdot(zu) = y\cdot(z - 2z^*) = y\cdot z - 2y\cdot z^* = m_1 + m_0 - 2y\cdot z^*$ and $2u\cdot(zy) = -2\alpha u + 2s_1 - a_{1/2}$. Thus $2(y\cdot z)u = 2y\cdot(zu) + 2u\cdot(zy) - 2(u\cdot z)y = -\alpha u + s_1 + m_1 + \beta v - s_0 + m_0 - 2y\cdot z^*$. But since y is in A_N it follows that $\alpha = \beta = 0$. Hence, $yz = t_1 + a_{1/2} + t_0$ and $zy = s_1 - a_{1/2} + s_0$ where t_i, s_i are in N_i.

Furthermore, $2m_1 = 2(y\cdot z)u = s_1 + m_1 - s_0 + m_0 - 2y\cdot z^*$ and since $y \in M_{1/2}$, so $y\cdot z^*$ is in $M_1 + M_0$, thus s_1 is in M_1 and s_0 is in M_0. Similarly t_1 is in M_1 and t_0 is in M_0.

Lemma 2.6. *For each x in A_N, and y in $A_{1/2}$, xy is in $N + A_{1/2}$, and yx is in $N + A_{1/2}$.*

Proof. As in Lemma 2.5, we need to show the following five statements.

(2.4.a) $$(A_NA_{1/2})_1 \subseteq N_1;$$

(2.4.b) $$(A_{1/2}A_N)_1 \subseteq N_1;$$

(2.4.c) $$(A_NA_{1/2})_0 \subseteq N_0;$$

(2.4.d) $$(A_{1/2}A_N)_0 \subseteq N_0;$$

(2.4.e) $$(A_NA_{1/2})_{1/2} = (A_{1/2}A_N)_{1/2}.$$

Again (2.4.e) is obvious. Let $z \in A_{1/2}$, then $2(y \cdot z)u = 2n_1$; $2(u \cdot z)y = zy = (zy)_1 + (zy)_{1/2} + (zy)_0$; $2y \cdot (zu) = y \cdot (z - 2z^*) = y \cdot z - 2y \cdot z^* = n_1 + n_0 - 2y \cdot z^*$; and $2u \cdot (zy) = 2(zy)_1 + (zy)_{1/2}$. Thus by (1.2) we have $2n_1 + (zy)_1 + (zy)_{1/2} + (zy)_0 = n_1 + n_0 - 2y \cdot z^* + 2(zy)_1 + (zy)_{1/2}$, hence $(zy)_1$ is in N_1 and $(zy)_0$ is in N_0. Note that y is in A_N so $y \cdot z^*$ is in $N = N_1 + N_0$.

Lemma 2.7. *If A_N is the singular subspace of $A_{1/2}$ then $N_1A_N \subseteq A_N$, and $N_0A_N \subseteq A_N$.*

Proof. Applying x in N_1, y in A_N and z in $A_{1/2}$ to the flexible identity $(xy) \cdot z = -(zy) \cdot x + x(y \cdot z) + z(y \cdot x)$, we find (zy) is in $A_{1/2}A_N \subseteq N + A_{1/2}$ by Lemma 2.6 so $(zy) \cdot x$ is in $N_1 + A_{1/2}$; $y \cdot z$ is in $N_1 + N_0$ by definition of y so $x(y \cdot z)$ is in N_1; and $y \cdot x$ is in A_N, $z(y \cdot x)$ is in $A_{1/2}A_N \subseteq A_{1/2} + N$ by Lemma 2.4 and Lemma 2.6. Thus $(xy) \cdot z$ is in $N + A_{1/2}$.

On the other hand, xy is in $A_{1/2}$,$(xy) \cdot z$ is in $A_1 + A_0$, hence $(xy) \cdot z$ is in N, or xy is in A_N.

Corollary. *The relations $CA_N = (A_1 + A_0)A_N \subseteq A_N$; $A_NN \subseteq A_N$ and $A_NC \subseteq A_N$ hold true.*

Lemma 2.8. *Let $J = M_1 + M_{1/2} + M_0$ be a proper A^+-ideal, then*
(a) $M_{1/2}N \subseteq M_{1/2} + NM_{1/2} \subseteq A_N$; (b) $M_{1/2}C \subseteq M_{1/2} + NM_{1/2} \subseteq A_N$; *and*
(c) $uA_N \subseteq A_N$.

Proof. (a) If x is in N, and y is in $M_{1/2}$, $yx = 2x \cdot y - xy$ is in $M_{1/2} + NM_{1/2} \subseteq A_N$, by Lemmas 2.4 and 2.7.

(b) $M_{1/2}C = M_{1/2} + M_{1/2}N \subseteq M_{1/2} + M_{1/2} + NM_{1/2} = M_{1/2} + NM_{1/2} \subseteq A_N$.

(c) If x is in $A_{1/2}$ and y is in A_N, $uy = \frac{1}{2}y + y^*, yu + \frac{1}{2}y - y^*$ so $x(uy) + y(ux) = \frac{1}{2}xy + xy^* + \frac{1}{2}yx + yx^*$, on the other hand, $(xu)y + (uy)x = \frac{1}{2}xy - x^*y + \frac{1}{2}yx - y^*x$. But $x(uy) + y(ux) = (xu)y + (yu)x$, thus $x \cdot y^* + y \cdot x^* = 0$, since y is in $A_n \cdot$ $y \cdot x^*$ is in N, that means $x \cdot y^*$ is in N, which in turn implies y^* is in A_N. Hence $uy = \frac{1}{2}y + y^*$ is in A_N.

Corollary 1. *If y is in $M_{1/2}$, then uy is in A_N; that is, $uM_{1/2} \subseteq A_N$.*

Corollary 2. *If y is in A_N and a is in A, then ay, ya both are in A_N.*

Proof. Let $a = \alpha u + n$ be in A_1. Then $ay = (\alpha u + n)y = \alpha(uy) + ny = \alpha(uy) + ny$ is in A_N by Lemma 2.7. and Lemma 2.8. Furthermore, since $y \cdot a = y \cdot (\alpha u + n) = (\alpha/2)y + y \cdot n$ is in $A_N + A_N$ by Lemma 2.4. Thus $ya = 2y \cdot a - ay$ is in A_N.

LEMMA 2.9. *If y is in $M_{1/2}$, then vy is in $M_{1/2} + uM_{1/2}$, that is*

$$vM_{1/2} \subseteq M_{1/2} + uM_{1/2}.$$

LEMMA 2.10. *Let $J = M_1 + M_{1/2} + M_0$ be a proper A^+-ideal, then*

$$A_{1/2}M_1 \subseteq A_N; \tag{2.5.a}$$

$$M_1A_{1/2} \subseteq A_N; \tag{2.5.b}$$

$$A_{1/2}M_0 \subseteq A_N; \tag{2.5.c}$$

$$A_{1/2}M_0 \subseteq A_N. \tag{2.5.d}$$

Proof. Let x, y be in $A_{1/2}$, m be in M_1, then $m \cdot (xy)$ is in $J = M_1 + M_{1/2} + M_0$; $(y \cdot x)m$ is in $(A_1 + A_0)M_1 \subseteq N_1$; and $(mx)y$ is in $M_{1/2}A_{1/2} \subseteq N + A_{1/2}$ by Lemma 2.5. Thus $y \cdot (xm) = -m \cdot (xy) + (y \cdot x)m + (m \cdot x)y$ is in $N + A_{1/2}$. But xm is in $A_{1/2}$, $y \cdot (xm)$ is in $A_1 + A_0$, thus $y \cdot (xm)$ is in N, or xm is in A_N as was to be proved for (2.5.a). Similarly for (2.5.c).

LEMMA 2.11. *Let $M_{1/2}$, $A_{1/2}$ be as mentioned in Lema 2.5; then*

$$\begin{aligned} (A_{1/2}M_{1/2})_{1/2} &= (M_{1/2}A_{1/2})_{1/2} \subseteq \bar{M}_{1/2} \\ &= \{y \text{ in } A_{1/2} \mid y \cdot z \text{ is in } M_1 + M_0 \text{ for all } z \text{ in } A_{1/2}\}. \end{aligned} \tag{2.6}$$

Proof. Let m be in $M_{1/2}$, and y, z be in $A_{1/2}$, then $(zy) \cdot m$ is in $J = M_1 + M_{1/2} + M_0$; $m(y \cdot z)$ is in $M_{1/2}(A_1 + A_0) \subseteq A_N$ by Corollary 2 of Lemma 2.8; and $z(y \cdot m)$ is in $A(M_1 + M_0) \subseteq A_N$, by Lemma 2.10. Thus $(my) \cdot z = -(zy) \cdot m + m(y \cdot z) + z(y \cdot m)$ is in $M_1 + A_N + M_0$. But since $(my) \cdot z = (my)_1 \cdot z + (my)_{1/2} \cdot z + (my)_0 \cdot z$ and $(my)_1 \cdot z + (my)_0 \cdot z$ is in $A_{1/2}$, $(my)_{1/2} \cdot z$ is in $M_1 + M_0$ follows. Thus $(M_{1/2}A_{1/2})_{1/2} \subseteq \bar{M}_{1/2}$. On the other hand, $(M \cdot y)_{1/2} = 0$ so $(my)_{1/2} = -(ym)_{1/2}$. Thus $(A_{1/2}M_{1/2})_{1/2} = (M_{1/2}A_{1/2})_{1/2}$.

LEMMA 2.12. *Let A_N be the singular subspace of $A_{1/2}$, then*

$$(A_NA_{1/2})_{1/2} = (A_{1/2}A_N)_{1/2} \subseteq A_N.$$

Proof. It suffices to show $(my)_{1/2}$ is in A_N if m is in A_N and y is in $A_{1/2}$. Let z be any element in $A_{1/2}$. We have: (a) $(zy) \cdot m = (zy)_1 \cdot m + (zy)_{1/2} \cdot m + (zy)_0 \cdot m$ is in $A_N + N + A_N = A_N + N$ by Lemma 2.4; (b) $m(y \cdot z)$ is in $A_N(A_1 + A_0) \subseteq A_N$; and (c) $A_{1/2}N \subseteq A_N$ by Lemma 2.4 and u-stability, thus $(my) \cdot z = -(zy) \cdot m + m(y \cdot z) + z(y \cdot m)$ is in $A_N + N$. On the other hand, $(my) \cdot z = (my)_1 \cdot z$

$+(my)_{1/2}\cdot z+(my)_0\cdot z$ is in $A_{1/2}$ where $(my)_1\cdot z+(my)_0\cdot z$ is in $A_{1/2}$ by u-stability, thus $(my)_{1/2}\cdot z$ is in N, whuch means $(my)_{1/2}$ is in $A_{1/2}$.

Now we define the following notations:

$$\begin{aligned} M_1^{(2)} &= M_1 + N_1M_1; \\ M_{1/2}^{(2)} &= M_1 + NM_{1/2} + M_1A_{1/2} + M_0A_{1/2} + uM_{1/2} \\ &\qquad + (A_{1/2}M_{1/2})_{1/2}; \\ M_0^{(2)} &= M_0 + N_0M_0; \\ J^{(2)} &= M_1^{(2)} + M_{1/2}^{(2)} + M_0^{(2)}. \end{aligned} \tag{2.7}$$

It is of interest to know that:

LEMMA 2.13. *$M_i^{(2)} \subseteq N_i$, $i=1,0$, and $M_{1/2}^{(2)} \subseteq A_N$.*

Proof. The first statement is obvious, since $M_i \subseteq N_i$ and the fact that N_i are subalgebras of A_i. The second half can be proved by Lemma 2.3, Corollaries to Lemma 2.7, Lemma 2.10, the Corollary to Lemma 2.11, and Lemma 2.10.

LEMMA 2.14. *Let J be a proper A^+-ideal and $M_i^{(2)}, J^{(2)}$ be defined as above, then, $AJ \subseteq J^{(2)}$ and $JA \subseteq J + AJ \subseteq J^{(2)}$.*

Proof. $AM_1 = A_1M_1 + A_{1/2}M_1 + A_0M_1 \subseteq M_1^{(2)} + M_{1/2}^{(2)} \subseteq J^{(2)}$. Similarly $AM_0 \subseteq M_0^{(2)} + M_{1/2}^{(2)} \subseteq J^{(2)}$, $AM_{1/2} = A_1M_{1/2} + A_{1/2}M_{1/2} + A_0M_{1/2}$, $+(A_{1/2}M_{1/2})_{1/2} + (A_{1/2}M_{1/2})_0 \subseteq M_{1/2}^{(2)} + M_1 + M_0 \subseteq J^{(2)}$ by the definition and Lemma 2.5. Thus $AJ \subseteq J^{(2)}$, moreover, $JA \subseteq J + AJ \subseteq J^{(2)}$, for J is an A^+-ideal.

LEMMA 2.15. *$M_1^{(2)}$ is an A_1^+-ideal, $M_0^{(2)}$ is an A^+-ideal.*

Proof. Since $M_1^{(2)} = M_1 + NM_1$, $A_1\cdot M_1^{(2)} = A_1\cdot(NM_1)$. However $A_1\cdot M_1 \subseteq M_1$, since J is an A^+-ideal, so it suffices to show $A_1\cdot(NM_1) \subseteq M_1^{(2)}$. Let n_1, and n_2 be in N, m be in M, then $m\cdot(n_2n_1)$ is in M_1, $(n_1\cdot n_2)m$ is in $NM_1 = N_1M_1$ and $(m\cdot n_2)n_1$ is in $M_1N \subseteq M_1 + N_1M_1$, by Lemma 2.3. Thus we have $n_1\cdot(n_2M) = -m\cdot(n_2n_1) + (n_1\cdot n_2)m + (m\cdot n_2)n_1$ is in $M_1 + N_1M_1 = M_1^{(2)}$.

LEMMA 2.16. *Let $J = M_1 + M_{1/2} + M_0$ be an A^+-ideal, then $A_{1/2}M_1 \subseteq M_{1/2} + M_1A_{1/2}$, and $A_{1/2}M_0 \subseteq M_{1/2} + M_0A_{1/2}$.*

LEMMA 2.17. *$M_1^{(2)}, M_{1/2}^{(2)}$ are subspaces defined by (2.7). Then*

$$M_1^{(2)}\cdot A_{1/2} \subseteq M_{1/2}^{(2)}.$$

Proof. We shall first show $(N_1M_1)\cdot A_{1/2} \subseteq M_{1/2}^{(2)}$. Let x be in $A_{1/2}$, n be in N_1 and m be in M_1, then $m\cdot(nx)$ is in $M_1\cdot A_{1/2} \subseteq M_{1/2} \subseteq M_{1/2}^{(2)}$, since J is an A^+-ideal; $x\cdot n$ is in $A_{1/2}$ and $(x\cdot n)m$ is in $M_1\cdot A_{1/2} \subseteq M_{1/2} + M_{1/2} \subseteq M_{1/2}^{(2)}$

by Lemma 2.15 and $(m\cdot n)x$ is in $M_1A_{1/2}\subseteq M_{1/2}^{(2)}$ since M_1 is an A_1^+-ideal. Thus, $x\cdot(nm) = -m\cdot(nx)+(x\cdot n)m+(m\cdot n)x$ is in $M_{1/2}^{(2)}$.

Obviously, $M_1\cdot A_{1/2}\subseteq M_{1/2}\subseteq M_{1/2}^{(2)}$. Thus

$$M^{(2)}\cdot A_{1/2} = M_i\cdot A_{1/2} + (N_1M_1)\cdot A_{1/2}\subseteq M_{1/2}^{(2)}.$$

This proves the lemma.

LEMMA 2.18. *Let $M_i^{(2)}$ be as defined by* (2.7). *Then*

$$M_1^{(2)}\cdot A\subseteq M_1^{(2)} + M_{1/2}^{(2)};$$

(2.8)

$$M_0^{(2)}\cdot A\subseteq M_{1/2}^{(2)} + M_0^{(2)}.$$

Proof. $M_1^{(2)}\cdot A = M_1^{(2)}\cdot A_1 + M_1^{(2)}\cdot A_{1/2} + M_1^{(2)}\cdot A_0\subseteq M_1{}^{(2)} + M_{1/2}^{(2)}$ follows from Lemma 2.15, Lemma 2.17 and the fact that $M_1^{(2)}\cdot A_0 = 0$. The second part follows similarly.

LEMMA 2.19. *Let $M_{1/2}^{(2)}$ be defined as above. Then $M_{1/2}^{(2)}\cdot A_1\subseteq M_{1/2}^{(2)}$.*

Proof. It suffices to show $M_{1/2}^{(2)}\cdot N_1\subseteq M_{1/2}^{(2)}$, for if y is in $M_{1/2}^{(2)}\subseteq A_{1/2}$, then $y\cdot u = \frac{1}{2}y$. We shall complete the proof by showing the following six relations.

(a) $M_{1/2}\cdot N_1\subseteq M_{1/2}^{(2)}$,

(b) $(NM_{1/2})\cdot N_1\subseteq M_{1/2}^{(2)}$,

(c) $(M_1A_{1/2})\cdot N_1\subseteq M_{1/2}^{(2)}$,

(d) $(M_0A_{1/2})\cdot N_1\subseteq M_{1/2}^{(2)}$,

(e) $(uM_{1/2})\cdot N_1\subseteq M_{1/2}^{(2)}$,

(f) $(A_{1/2}M_{1/2})_{1/2}\cdot N_1\subseteq M_{1/2}^{(2)}$.

(a) $M_{1/2}\cdot N_1\subseteq M_{1/2}$, since J is an A^+-ideal.

(b) Let n be in N_1, a be in N, and m be in $M_{1/2}$, then $m\cdot(an)$ is in $M_{1/2}\cdot N_1\subseteq M_{1/2}\subseteq M_{1/2}^{(2)}$, since J is an A^+-ideal, $(n\cdot a)m\subseteq N_1M_{1/2}\subseteq M_{1/2}^{(2)}$ by definition, and $(m\cdot a)n$ is in $M_{1/2}N_1\subseteq M_{1/2}^{(2)}$. Thus $n\cdot(am) = -m\cdot(an)+(n\cdot a)m + (m\cdot a)n\subseteq M_{1/2}^{(2)}$. Hence $N_1\cdot(NM_{1/2})\subseteq M_{1/2}^{(2)}$.

(c) Let n be in N_1, m be in M_1, x be in $A_{1/2}$, then $x\cdot(mn)$ is in $A_{1/2}\cdot(M_1N_1)\subseteq M^{(2)}$ by Lemma 2.17, $(n\cdot m)x$ is in $M_1A_{1/2}\subseteq M_{1/2}^{(2)}$ by definition of $M_{1/2}^{(2)}$, and $(x\cdot m)n$ is in $M_{1/2}N_1\subset M_{1/2}+N_1M_{1/2}\subset M_{1/2}^{(2)}$. Thus $n\cdot(mx) = -x\cdot(mn)+(n\cdot m)x+(x\cdot m)n$ in $M_{1/2}^{(2)}$. Hence $N\cdot(M_1A_{1/2})\subseteq M_{1/2}^{(2)}$.

(d) Let n be in N_1, m be in M_0, and x be in $A_{1/2}$. We have $n\cdot(mx) = (x\cdot m)n$ contained in $M_{1/2}N_1\subseteq M_{1/2}+N_1M_{1/2}\subseteq M_{1/2}^{(2)}$, noted that $nm = n\cdot m = 0$.

(e) Let x be in $M_{1/2}$, y be in N_1, then $uy = y$ is in N_1 so $x\cdot(uy) = x\cdot y$ is in $M_{1/2}\subseteq M_{1/2}^{(2)}$, $(y\cdot u)x = yx$ is in $N_1M_{1/2}\subseteq M_{1/2}^{(2)}$, and $(x\cdot u)y = \frac{1}{2}xy$ is in $M_{1/2}N\subseteq M_{1/2}+NM_{1/2}\subseteq M_{1/2}^{(2)}$. Hence we have $y\cdot(ux) = -x\cdot(uy)+(y\cdot u)x + (x\cdot u)y$ contained in $M_{1/2}^{(2)}$.

(f) Let a be in $A_{1/2}$, m be in $M_{1/2}$, and n be in N_1 by the Corollaries of Lemma 2.7, nm is in A_N and so $(nm)\cdot a$ is in N. Also $a(m\cdot n)$ is in

$A_{1/2}M_{1/2} \subseteq M_1 + (A_{1/2}M_{1/2})_{1/2} + M_0$ by Lemma 2.5, and $n(m\cdot a)$ is in $N_1(M_1+M_0) \subseteq M_1^{(2)}$, thus $(am)\cdot n = -(nm)\cdot a + a(m\cdot n) + n(m\cdot a)$ is in $N + (A_{1/2}M_{1/2})_{1/2}$. But $(am)\cdot n = (am)_1\cdot n + (am)_{1/2}\cdot n + (am)_0\cdot n$, and considering that $(am)_1\cdot n$ is in N_1, and $(am)_0\cdot n = 0$, one gets $(am)_{1/2}\cdot n$ is in $(A_{1/2}M_{1/2})_{1/2}$.

COROLLARY. *Let $M_{1/2}^{(2)}$ defined as above. Then $M_{1/2}^{(2)}\cdot A_0 \subseteq M_{1/2}^{(2)}$.*

LEMMA 2.20. *Let $M_{1/2}^{(2)}$ defined as above. Then*

$$M_{1/2}^{(2)}\cdot A_{1/2} \subseteq M_1^{(2)} + M_0^{(2)}.$$

Proof. It suffices to show following six relations for the completion of the proof of the lemma.

(2.9.a) $M_{1/2}\cdot A_{1/2} \subseteq M_1^{(2)} + M_0^{(2)}$;
(2.9.b) $(NM_{1/2})\cdot A_{1/2} \subseteq M_1^{(2)} + M_0^{(2)}$;
(2.9.c) $(M_1A_{1/2})\cdot A_{1/2} \subseteq M_1^{(2)} + M_0^{(2)}$;
(2.9.d) $(M_0M_{1/2})\cdot A_{1/2} \subseteq M_1^{(2)} + M_0^{(2)}$;
(2.9.e) $(uM_{1/2})\cdot A_{1/2} \subseteq M_1^{(2)} + M_0^{(2)}$;
(2.9.f) $(A_{1/2}M_{1/2})_{1/2}\cdot A_{1/2} \subseteq M_1^{(2)} + M_0^{(2)}$.

(a) Obvious $M_{1/2}\cdot A_{1/2} \subseteq M_1 + M_0 \subseteq M_1^{(2)} + M_0^{(2)}$, since J is an A^+-ideal.

(b) Let n be in N, x be in $M_{1/2}$, and y be in $A_{1/2}$, then ny is in $NA_{1/2} \subseteq A_{1/2}$, so $x\cdot(ny)$ is in $M_{1/2}\cdot A_{1/2} \subseteq M_1 + M_0$, $(y\cdot n)x$ is in $A_{1/2}M_{1/2} \subseteq M_1 + M_0 + A_N$ by Lemma 2.5 and Lemma 2.11, and $(x\cdot n)$ is in $N\cdot M_{1/2}$, so $(x\cdot n)y$ is in $M_{1/2}A_{1/2} \subseteq M_1 + M_0 + A_N$ by Lemma 2.6 and Lemma 2.11. Thus $y\cdot(nx) = -x\cdot(ny) + (y\cdot n)x + (x\cdot n)y$ is in $M_1 + M_0 + A_N$. But y is in $A_{1/2}$, n is in N, x is in $M_{1/2}$, $y\cdot nx$ is in $A_{1/2}\cdot A_{1/2} \subseteq A_1 + A_0$ and so $v\cdot(nx)$ is in $M_1 + M_0$.

(c) Let m be in M_1, x, y be in $A_{1/2}$, then $(yx)\cdot m$ is in $M_1 + M_{1/2} + M_0 \subseteq M_1^{(2)} + M_0^{(2)} + A_N$. Since m is in M_1 and J is an A^+-ideal, $m(x\cdot y)$ is in $M_1A_1 \subseteq M_1^{(2)}$ by definition and, $y(x\cdot m)$ is in $M_{1/2}A_{1/2} \subseteq M_1 + M_0 + A_N$ by Lemma 2.5 and Lemma 2.12. Thus $(mx)\cdot y = -(yx)\cdot m + m(x\cdot y) + y(x\cdot m)$ is in $M_1^{(2)} + M_0^{(2)} + A_N$. But $(mx)\cdot y$ is in $A_{1/2}\cdot A_{1/2} \subseteq A_1 + A_0$, hence $(mx)\cdot y$ is in $M_1^{(2)} + M_0^{(2)}$.

(d) Similar to (c).

(e) Let x be in $M_{1/2}$, and y be in $A_{1/2}$, by (1.2)

$$\begin{aligned} y\cdot(ux) &= -x\cdot(uy) + (y\cdot u)x + (x\cdot u)y \\ &= -x\cdot(uy) + 1/2\,yx + 1/2\,xy \\ &= -x\cdot(uy) + x\cdot y \\ &= x\cdot(y - uy) \\ &= x\cdot(yu) \text{ is in } M_1 + M_0. \end{aligned}$$

(f) Let x,y be in $A_{1/2}$, and m be in $M_{1/2}$, obvious $(yx)\cdot m$ is in $M_1+M_{1/2}+M_0, m(x\cdot y)$ is in $M_{1/2}(A_1+A_0)\subseteq M_{1/2}+A_1M_{1/2}+A_0M_{1/2}\subseteq M_{1/2}^{(2)}$ by Lemma 2.8 and by definition of $M_{1/2}^{(2)}$, and $y(x\cdot m)$ is in $A_{1/2}(M_1+M_0)$ $\subseteq M_{1/2}+M_1A_{1/2}+M_0A_{1/2}\subseteq M_{1/2}^{(2)}$ by the fact that J is A^+-ideal and $A_{1/2}M_1\subseteq M^{(2)}$. Thus $(mx)\cdot y = -(yx)\cdot m + m(x\cdot y) + y(x\cdot m)$ is in $M_1 + M_{1/2}^{(2)} + M_0 \cdot (mx)\cdot y = (mx)_1\cdot y + (mx)_{1/2}\cdot y + (mx)_0\cdot y$. Thus $(mx)_1\cdot y+(mx)_0\cdot y$ is in $A_{1/2}$ so $(mx)_{1/2}\cdot y$ is in M_1+M_0.

THEOREM 2.2. *$J^{(2)}$ is an A^+-ideal containing J as a subideal of $J^{(2)}$. Furthermore, J is either a proper subideal of $J^{(2)}$ or J is an A-ideal.*

Proof. The first statement is a consequence of Lemmas 2.18, 2.19 2.20 and by the definition of $J^{(2)}$. Obvious $J\subseteq J^{(2)}$, if $J=J^{(2)}$, then $AJ\subseteq J^{(2)}=J$, by Lemma 2.14, similarly, $JA\subseteq J^{(2)}=J$, thus J is an A-ideal.

LEMMA 2.21. *$J^{(2)}$ is properly contained in A.*

Proof. By definition of $M_1^{(2)}$ and $M_0^{(2)}$ and by Lemma 2.13, u is not in $M_1^{(2)}$ and v not in $M_0^{(2)}$, so we have, in fact, $J^{(2)}\subseteq N+A_{1/2}$.

LEMMA 2.22. *Let $M_{1/2}^{(2)}$ be as defined in* (2.7). *Then $M_{1/2}^{(2)}\subseteq A_N$.*

Proof. Since $J^{(2)}$ is a proper A^+-ideal by Theorem 2.2, it follows from Lemma 2.3 that $M_{1/2}^{(2)}\subseteq A_N$.

Let us now summarize the properties of $J^{(2)}$. So far we have shown that $J^{(2)}=M_1^{(2)}+M_{1/2}^{(2)}+M_0^{(2)}$ is an A^+-ideal, and is properly contained in A^+-ideal, and is properly contained in A^+. Furthermore, $J^{(2)}$ properly contains J if J is not an A-ideal (Lemma 2.21). Finally, $J^{(2)}=M_1^{(2)}+M_{1/2}^{(2)}+M_0^{(2)}$ where $M_i^{(2)}\subseteq N_i$, $i=0,1$, and $M_{1/2}^{(2)}\subseteq A_N$ as was proved in Lemma 2.13 and Lemma 2.22. $J^{(2)}$plays the same role in this chapter that J does, we may recursively define the following terms:

$$(2.10)\qquad \begin{aligned} M_1^{(1)} &= M_1;\\ M_{1/2}^{(1)} &= M_{1/2};\\ M_0^{(1)} &= M_0;\\ J^{(1)} &= M_1^{(1)}+M_{1/2}^{(1)}+M_0^{(1)}.\end{aligned}$$

$$(2.11)\qquad \begin{aligned} M_1^{(i+1)} &= M_1^{(i)}+N_1M_1^{(i)};\\ M_{1/2}^{(i+1)} &= M_{1/2}^{(i)}+NM_{1/2}^{(i)}+M_1^{(i)}A_{1/2}+uM_{1/2}^{(i)}+(A_{1/2}M_{1/2}^{(i)})_{1/2};\\ M_0^{(i+1)} &= M_0^{(i)}+N_0M_0^{(i)};\\ J^{(i+1)} &= M_1^{(i+1)}+M_{1/2}^{(i+1)}+M_0^{(i+1)}.\end{aligned}$$

$J^{(1)}$ is just the A^+-ideal J from which our discussion started.

$J^{(2)}$ is the A^+-ideal mentioned in Theorem 2.2, which properly contains J and $J^{(2)}$ and itself is contained in $N + A_N$ (Lemma 2.21). Along this line, we define $J^{(3)}$ from $J^{(2)}$, and the same conclusion can be obtained. Then $J^{(3)}$ is an A^+-ideal or $J^{(3)}$ properly contains $J^{(2)}$, since simplicity of A is assumed. Hence, $J^{(2)} = 0$ or $J^{(3)} \supset J^{(2)}$, $J^{(3)} \neq J^{(2)}$. However, if $J^{(2)} = 0$, then $J^{(1)} = J = 0$, and we have proved what we want to show; otherwise $J^{(3)} \supset J^{(2)}$, $J^{(3)} \neq J^{(2)}$, $J^{(3)}$ properly contains $J^{(2)}$. We stay on the same track and obtain a sequence of strictly increasing sequence of A^+-ideals, $J^{(1)} \subseteq J^{(2)} \subseteq J^{(3)} \subseteq \cdots \subseteq J^{(k)} \subseteq J^{(k+1)} \subseteq \cdots$, none of $J^{(i)} = A^+$. But, A^+ is of finite dimension, so this sequence must have finite length and we have

THEOREM 2.3. *Let $J^{(1)}, J^{(2)}, \cdots$ be the ideals obtained above, then there exists a positive integer k such that $J^{(k)} = J^{(k+1)}$, and furthermore, $J^{(k)}$ is an A-ideal.*

Proof. The first part of the theorem follows since A^+ is finite dimensional while the second part follows from the definition of $J^{(k+1)}$ and Lemma 2.5.

Now, we can go back and complete the proof of Theorem 2.1. We start with an A^+-ideal J, which is not equal to A^+. If $J \neq 0$, we construct $J^{(1)}, J^{(2)}, \cdots, J^{(k)}$, an increasing sequence of A^+-ideals, and by Theorem 2.3, we have $J \subseteq J^{(k)} = J^{(k+1)}$, and $J \subseteq J^{(k)} = 0$, which leads to a contradiction. On the other hand, if J is an A^+-ideal, such that $J \neq A^+$, then $J = 0$. This means A^+ contains no proper ideal which is exactly what we want to show, i.e. A^+ is simple. We have now proved Theorem 2.1.

III. **The structure of** A. Since for every simple, flexible power associative algebra A over an algebraically closed field F of characteristic not 2, 3 or 5 the associated commutative algebra A^+ is also simple. The algebra A is an algebra of the type in [4], [5].

A flexible algebra over a field F of characteristic prime to 30 is power associative if, and only if, $x^2x^2 = (x^2x)x$ for every x in A, which in turn means if, and only if, the equation (3.1) below is satisfied by every element x, y, z, w in A.

$$\begin{aligned}
&(xy + yz)(zw + wz) + (zw + wz)(xy + yx) + (xz + zx)(yw + wy) \\
&\quad + (yw + wy)(xz + zx) + (xw + wx)(yz + zy) + (yz + zy)(xw + wx) \\
&\quad = [(yz + zy)w + (zw + wz)y + (yw + wy)z]x + [(xz + zx)w + (xw + wx)z \\
&\quad + (wz + zw)x]y + [(xy + yx)w + (yw + wy)x + (xw + wx)y]z \\
&\quad + [(xy + yx)z + (yz + zy)x + (xz + zx)y]w .
\end{aligned} \tag{3.1}$$

An element y in $A_{1/2}$ is called a nonsingular element if $y^2 = \alpha + z$, where z is nilpotent and α is in F and not equal to zero.

If A is simple and u-stable, then there is a nonsingular element y in $A_{1/2}$. There exists an x in $F(y^2)$, the algebra generated by y^2 over F, such that $w = x \cdot y$ is in $A_{1/2}$ and $w^2 = 1$. (See [3], [4], [5], [6] and [12].)

LEMMA 3.1. *Let B be the set of all elements b of $C = A_1 + A_0$, such that $b = (b \cdot w) \cdot w$. Then, B^+ is a subalgebra of C^+, both A_1^+ and A_0^+ are isomorphic to B^+ and $A_1^+ = u \cdot B$, $A_0^+ = v \cdot B$, $C^+ = B^+ + z \cdot B^+$. Furthermore, $B^+ = F(x_1, x_2, \cdots, x_r)$ is the truncated polynomial algebra generated by r variables.*

The proof of the Lemma is given in [3] and [12]. It was also proved that if a and b are in B, then $(w \cdot a) \cdot b = (w \cdot b) \cdot a = w \cdot (a \cdot b)$, $(w \cdot a) \cdot (w \cdot b) = a \cdot b$ and $w \cdot (B^+ \cdot z) = 0$ and $w \cdot (a \cdot u) = w \cdot (a \cdot v) = 1/2(w \cdot a)$. It was also shown that the subspace $A_{1/2}$ can be decomposed as $A_{1/2} = w \cdot B + G$, where G consists of element g of $A_{1/2}$ such that $w \cdot g = 0$.

For every c in C we have $w(w \cdot c) = (w \cdot c)w$ in B. In particular, if c is in B, then $w(w \cdot c) = (w \cdot c)w = c$. If x is in A_1 and w is in $A_{1/2}$, then $u(wx) = (xw)u$. If x is in A_0, then $u(xw) = (wx)u$. Also, if c is in A_1, then, $2((c \cdot w)w)_1 = c$. If b is in B, then, $2((b_0 \cdot w)w)_1 = b_1 = 2((b_1 \cdot w)w)_1$.

We shall develop further some structure properties of A.

LEMMA 3.2. *The subspace $A_{1/2}$ can be represented as:*

$$A_{1/2} = L = (y_0 \cdot B,\ y_1 \cdot B, \cdots, y_m \cdot B), \tag{3.2}$$

where y_i's are $m + 1$ linear independent elements in $A_{1/2}$. If a and b are in B, then, $(y_i \cdot a) \cdot b = b \cdot (y_i \cdot a) = y_i \cdot (a \cdot b)$, and $(u \cdot a) \cdot (y_i \cdot b) = (v \cdot a) \cdot (y_i \cdot b) = \frac{1}{2} y_i \cdot (a \cdot b)$.

Proof. The subspace $A_{1/2}$ is same as $A_{1/2}^+$. (3.2) is given in [4].

LEMMA 3.3. *For $i = 0, 1, \cdots, m$ and an arbitrary element x in B, it follows that*

$$xy_i = y_i x = y_i \cdot x. \tag{3.3}$$

Proof. Applying (3.1), we let $x = x$ in B, $y = y_i$ in $A_{1/2}$, $z = u$, and $w = v$. We make use of the orthogonality of A_1 and A_0 and of the fact $ua = au = a_1$ and $va = av = a_0$, where $a = a_1 + a_0$, for every a in C. We find:

$$\begin{aligned} 2x_1 y_i + 2y_i x_1 + 2x_0 y_i + 2y_i x_0 &= (y_i u + y_i v)x \\ &+ ((xy_i + y_i x)v + y_i x + 2x_0 y_i)u \\ &+ ((xy_i + y_i x)u + y_i x + 2x_1 y_i)v. \end{aligned}$$

Continuing to simplify, consider $x_0 = xv = vx$, $x_1 = xu = ux$, we have

$$\begin{aligned} 2xy_i &= ((xy_i + y_ix)v + 2x_0y_i)u + ((xy_i + y_ix)u + 2x_1y_i)v \\ &= ((xy_i)v + (y_ix)v + (xv)y_i + (vx)y_i)u \\ &\qquad + ((xy_i)u + (y_ix)u + (xu)y_i + (ux)y_i)v \\ &= ((xy_i)v + (xv)y_i + y_i(xv) + v(xy_i))u \\ &\qquad + ((xy_i)u + (xu)y_i + y_i(xu) + u(xy_i))v \\ &= 2((xy_i)\cdot v + x_0\cdot y_i)u + 2((xy_i)\cdot u + x_1\cdot y_i)v \\ &= xy_i + 2(x_0\cdot y_i)u + 2(x_1\cdot y_i)v \\ &= xy_i + 2[(v\cdot x)\cdot y_i]u + 2[(u\cdot x)\cdot y_i]v \\ &= xy_i + (y_i\cdot x)u + (y_i\cdot x)v \\ &= xy_i + y_i\cdot x \end{aligned}$$

by using the flexible identity (1.4) and Lemma 3.2.

Hence, $xy_i = y_ix = x\cdot y_i = y_i\cdot x$.

LEMMA 3.4. *For $i = 0, 1, \cdots, m$ and arbitrary elements a and b in B, then the following equations hold.*

$$a(y_ib) = (y_ib)a = a\cdot(y_ib) \tag{3.4}$$

and

$$a(y_ib) = a\cdot(y_ib) = a\cdot(y_i\cdot b) = y_i\cdot(a\cdot b). \tag{3.5}$$

Proof. Relation (3.5) follows directly from (3.4), Lemma 3.2 and Lemma 3.3, so we need only to show (3.4). For this purpose we apply (3.1) letting $x = a$, $y = y_ib$, $z = u$ and $w = v$. Similar to that in Lemma 3.3, we find

$$\begin{aligned} 2ay &= [(ay + ya)v + 2a_0y]u + [(ay + ya) + 2a_1y]v \\ &= [(ay)v + (av)y + (ya)v + (va)y]u + [(ay)u + (au)y + (ya)u + (ya)y]v \\ &= [(ay)v + (av)y + y(av) + v(ay)]u + [(ay)u + (au)y + y(au) + u(ay)]v \\ &= 2(v\cdot(ay) + a_0\cdot y)u + 2(u\cdot(ay) + a_1\cdot y)v, \end{aligned}$$

by the flexible identity.

Considering ay in $A_{1/2}$, we have $u\cdot(ay) = v\cdot(ay) = \frac{1}{2}ay$, and also $a_0\cdot y = a_0\cdot(y_i\cdot b) = \frac{1}{2}y_i\cdot(a\cdot b) = \frac{1}{2}a\cdot(y_i\cdot b)$ by Lemma 3.2. Hence,

$$2a(y_ib) = 2[a(y_ib) + a\cdot(y_ib)]u + [a(y_ib) + a\cdot(y_ib)]v$$

and

$$2a(y_ib) = a(y_ib) + a\cdot(y_ib) \text{ or } a(y_ib) = (y_ib)a = a\cdot(y_ib),$$

which is what we wanted to show.

THEOREM 3.1. *Every element in B commutes with every element in $A_{1/2}$.*

Proof. If z is in $A_{1/2}$, then, by Lemma 3.2, $z = y_0x_0 + y_1x_1 + \cdots + y_mx_m$, where the x_i's are in B. Hence for every a in B, $az = a(y_0x_0 + y_1x_1 + \cdots + y_mx_m) = (y_0x_0 + y_1x_1 + \cdots + y_mx_m)a = za$ by Lemma 3.4.

LEMMA 3.5. *If a and b are elements of B, then $b(aw) = w\cdot(ba)$ and $(wa)b = w\cdot(ab)$.*

Proof. Since w is in $A_{1/2}$ and B^+ is a subalgebra of A^+, by Theorem 3.1 $wa = aw = w\cdot a$, and $2(a\cdot b)\cdot w = 2(a\cdot b)w = 2w(a\cdot b)$, but $2(a\cdot b)\cdot w = 2b(a\cdot w)$ by Theorem 3.1. Combining these results, we have $w(ab) + w(ba) = b(aw) + b(wa)$. Since by the flexible identity (1.4) $(wa)b + (ba)w = w(ab) + b(aw)$, we have $w\cdot(ba) = b(aw) = (wa)b$.

It is also known that B is a subalgebra of A and both A_1 and A_0 are isomorphic algebras of B. (See [12].) In fact:

THEOREM 3.2. *The subalgebra B is a commutative associative subalgebra of A. In fact $B \cong B^+$, and thus $A_1 \cong A_1^+$ and $A_0 \cong A_0^+$.*

Proof. We need only to show this for arbitrary elements a and b in B. Toward this end, we see that w is in $A_{1/2}$, so wa, aw are in $A_{1/2}$ by u-stability of A. Then, by Theorem 3.1 and Lemma 3.5, $w\cdot(ba) = b(aw) = (aw)b = (wa)b = w\cdot(ab)$, and we have $w\cdot(ba) = w\cdot(ab)$. However, ab and ba are both in B by Lemma 3.6, hence, $w\cdot(w\cdot(ba)) = w\cdot(w\cdot(ab))$ or $ab = ba$, which is what we wanted to show.

A linear transformation of an algebra B is called a derivation on B if it satisfies the identity $(xy)D = (xD)y + x(yD)$ for all x, y in B.

It is easy to show that every derivation of B is also a derivation of the attached algebra B^+. However, in general, a derivation of B^+ is not necessarily a derivation of B. In the special case we discuss in this chapter, the algebra B is a commutative subalgebra of A, hence $B = B^+$, and every derivation of B^+ is a derivation of B. An ideal, J of B, is called a D-ideal, if $JD \subseteq J$, where JD designates the image set of J under the derivation D. An algebra B is called D-simple if B does not contain a proper D-ideal. Let $\mathscr{D}(D)$ be a set of derivations of B, then B is called $\mathscr{D}$-simple if B is D-simple with respect to every derivation D in $\mathscr{D}(D)$.

Applying the result of A. Albert and L. Harper [4] and [6] to the subalgebra B of A, we may state

THEOREM 3.3. *Let A be a simple, flexible, u-stable, power associative algebra of degree two over an algebraically closed field F of characteristic not equal to 2, 3 and 5. Then the subalgebra B is $\mathscr{D}$-simple with respect to a set of $(m+1)^2$ derivations of B, subject to the condition that*

$$D_{ij} = -D_{ji}, \qquad D_{ii} = 0, \qquad i,j = 0,1,\cdots,m. \tag{3.6}$$

Furthermore, $B = F(x_1,\cdots,x_r)$ is a truncated polynomial algebra of r generators over F such that $x_i^0 = 1$, and $x_i^p = 0$ for $i = 1,2,\cdots,r$, and p is the characteristic of F.

LEMMA 3.7. *Let $y_i, i = 1,2,\cdots,m$, be as defined in Lemma 3.2 and a be any element in B. Then*

$$\begin{aligned} (y_ia)u &= (y_iu)a\,; \\ u(y_ia) &= (uy_i)a\,. \end{aligned} \tag{3.7}$$

Proof. Applying the flexible identity (1.2), we have $(a\cdot y_i)u + (u\cdot y_i)a = a\cdot(y_iu) + u\cdot(y_ia)$. Since both y_i and y_ia are in $A_{1/2}$, $u\cdot y_i = \frac{1}{2}y_i$ and $u\cdot(y_ia) = \frac{1}{2}y_ia$. Furthermore, since a is in B, $a\cdot y_i = ay_i = y_ia$ by Lemma 3.3. Thus $(y_ia)u = (y_iu)a$. Similarly $u(y_ia) = (uy_i)a$.

LEMMA 3.8. *Let y_i, $i = 1,\cdots,m$, and a be defined as in the previous lemma. Then*

$$\begin{aligned} u(y_ia) &= (ua)y_i = a_1y_i\,; \\ (y_ia)u &= y_i(ua) = y_ia_1\,. \end{aligned} \tag{3.8}$$

Proof. By the flexible identity (1.1b), we have $y_i(ua) + a(uy_i) = (y_iu)a + (au)y_i$ or

$$y_ia_1 - a_1y_i = (y_iu)a - a(uy_i) = (y_ia)u - u(y_ia) \tag{3.9}$$

by Lemma 3.7. On the other hand, since $y_i\cdot a_1 = y_1\cdot(u\cdot a) = \frac{1}{2}y_i\cdot a = \frac{1}{2}y_ia$ by Lemma 3.2, $y_ia_1 = 2y_i\cdot a_1 - a_1y_i = y_ia - a_1y_i$. Thus

$$y_ia_1 + a_1y_i = y_ia = (y_ia)u + u(y_ia). \tag{3.10}$$

Adding (3.9) to (3.10), we have $y_ia_1 = (y_ia)u$. Similarly, $a_1y_i = u(y_iu)$.

LEMMA 3.9. *Let y_i, $i = 1,\cdots,m$ be defined as in Lemma 3.2, if $uy_i = \frac{1}{2}y_i$, then $ux = \frac{1}{2}x = xu$ for all x in $A_{1/2}$.*

Proof. Let $x = y_0a_0 + y_1a_1 + \cdots + y_ma_m$, a_i in B. Then

$$\begin{aligned} ux &= u(y_0a_0 + \cdots + y_ma_m) = (uy_0)a_0 + \cdots + (uy_m)a_m \\ &= \tfrac{1}{2}[y_0a_0 + \cdots + y_ma_m] = \tfrac{1}{2}x\,. \end{aligned}$$

Furthermore, $xu = ux = x - ux = x - \frac{1}{2}x = \frac{1}{2}x$. We have proved the lemma.

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